



The New Hybridized Model for the Standard Pooling Problem

Meninwa C.A^{1*}, Osagiede A.A²

^{1, 2} Mathematics Department, University of Benin, Benin City.

*Corresponding Author

Meninwa C.A

Mathematics Department,
University of Benin, Benin City.

Article History

Received: 09.01.2024

Accepted: 27.01.2024

Published: 06.02.2024

Abstract: The term "pooling problem" describes a situation in which a variety of products are blended together in a set of pools so that the combined products' qualities must meet specific standards before they can be transported downstream or to the intended audience when needed. Several models and approaches to solving the related blending & transportation process issues have been developed. We specifically recognized the difficulties caused by certain contemporary models' inability to process big dataset instances in a fair amount of time, as well as the difficulties associated with enhancing the lower bound of the relaxation solution methods utilized and optimizing product quality.

Thus, we present a new hybridized reformulation (NHYB⁻) and its associated solution method to handle the above challenges. We also considered some Mathematical properties of the formulated model. To illustrate the flexibility, robustness and smart nature of the proposed new hybridized models (NHYB⁻), the simulation study to generating large instances of dataset was carried out. Again, the applications of the simulated and lifetime data from the literature to the new hybridized reformulation of standard pooling problems were carried out with a few graphical illustrations

Results showed that our formulated NHYB⁻ performed greedily over the existing ones. By implication, our model demonstrates strengths in the area of time management, optimal solution and tenacity to handle larger instances which others could not handle.

Keywords: global solution methods, local solution methods, pooling problems, relaxation.

1. INTRODUCTION

Pooling problem indicates a generic study in the realm of mathematical optimization and a fundamental problem in the petroleum industry. The pooling problem involves the planning issue that occurs when materials are combined to create products, such as when combining crude and refined petroleum. Many challenges are encountered in the extraction, refining and transportation processes of petroleum products. In the refining and transportation processes, blending or pooling are needed as one of the means of obtaining the final product needed by the consumers. Hence, pooling problem which is a resultant process of the petroleum industry has become an area of attraction in the recent time. By virtue of this, many researchers have studied this area because of its vital application to the petroleum industry and other related fields. Two things that come to mind in the study of pooling problem are the minimization of cost of transportation and the improvement of quality of products at the destined areas for consumers.

The whole process is aimed at realizing the correct mixing proportions so as to minimize the general cost of transporting the product to the needed destination. The pooling issue is not exclusive to the oil industry; similar issues can arise with the manufacture of gas, chemicals, drinks, or even food wherever that a network of two or more sources might combine materials with varying degrees of quality- Lennart et al (2012). Because the flow

rates, which are also unknown, and the quality of the source or input streams rely on each other, the pooling problem becomes a bilinear optimisation problem - Armagan, (2009). Pooling brings nonlinearities and non-convexities into optimisation models as a result of the bilinear terms, which can result in several locally optimum solutions, some of which may be suboptimal.

The origin of pooling problems is traceable to the work of Haverly (1978) where he developed the P-formulation (flow formulation). For a large number of topological instantiations of the pooling issue, the 'Q'-formulation, which was first considered by Bental et al. (1994) and has a reduced problem size, was developed. By adding a cut to the 'Q'-formulation's constraint set, Quesada and Grossmann (1995) and Tawarmalani and Sahinidis (2002) created the 'PQ'-formulation, a third formulation of the pooling problem that combines the advantageous characteristics of a small problem formulation with a tight relaxation. A variant of the standard problem, where the total flow into each output is fixed to a nonzero constant, was examined by (Ruiz et al., 2013). Misener et al. (2010) introduced an improved pooling problem that establishes upper bounds on output-related emissions.

Additional examples of MINLP models are given by D'Ambrosio et al. (2011), Meyer and Floudas (2006), Misener and Floudas (2010), Mostafa (2010), Nishi (2010) and Visweswaran (2009). These MINLP variants are mostly caused by adding binary choice variables linked to the use of each arc or node in the graph, or by

forcing the flows to be semi continuous. Alfaki and Haugland (2011) suggested a method that discretizes the domain of the proportion variables into a predetermined number of points and linearizes the bilinear terms; this model approximates the pooling problem. Alfaki and Haugland (2013) also considered the strong formulations where the source and terminal proportions were combined to give the sophisticated results as presented in their work. Radu and Misener(2018) studied the piecewise parametric structure in the pooling problem: from sparse strongly-polynomial solutions to NP- hardness.

As a build up to the discretization methods of solving the pooling problem, Chen et al (2022), developed a tightening constraints model for the MILP; this model considers the MILP model by using the upper bounds on the bilinear terms. Rounding cuts for flows into the pools and dependent upper bounds on flows into the pools were employed to achieve their aim in the study. Michal and Rafal (2023) applied the pooling problem to social network analysis and mining where network structures of urban ride and their rides were considered. It was investigated that at the stage of feasible rides and identifying the optimal solution, clear distinction between simple and exact representations were made. They considered the deterministic and probabilistic networks. By

extension, they introduced the weighted networks with its corresponding properties including the properties of the other two networks.

In the next section, we will consider the expedient establishment of our proposed model and underlying mathematical properties of this model. The models tend to offer some complexities in computation as a result of the large instances it is intended to address cum other qualities. We started by illustrating the network of flow of materials through a schematic diagram of a standard pooling problem (SPP) and proposal of the new hybridized model (NHYB) followed. It is worthy of note that the new established model has its associated solution method. The linear relaxation approach which opens up for the search and branching tree was employed. Here also, we mirrored the concave and the convex envelopes arising from the relaxation solution method. We also explored some mathematical properties of the model.

2. METHODOLOGY

Consider the diagram below with the defined network structure. Our motivation for the perspective of this work begins here and it succinctly explains the rationale behind this study.

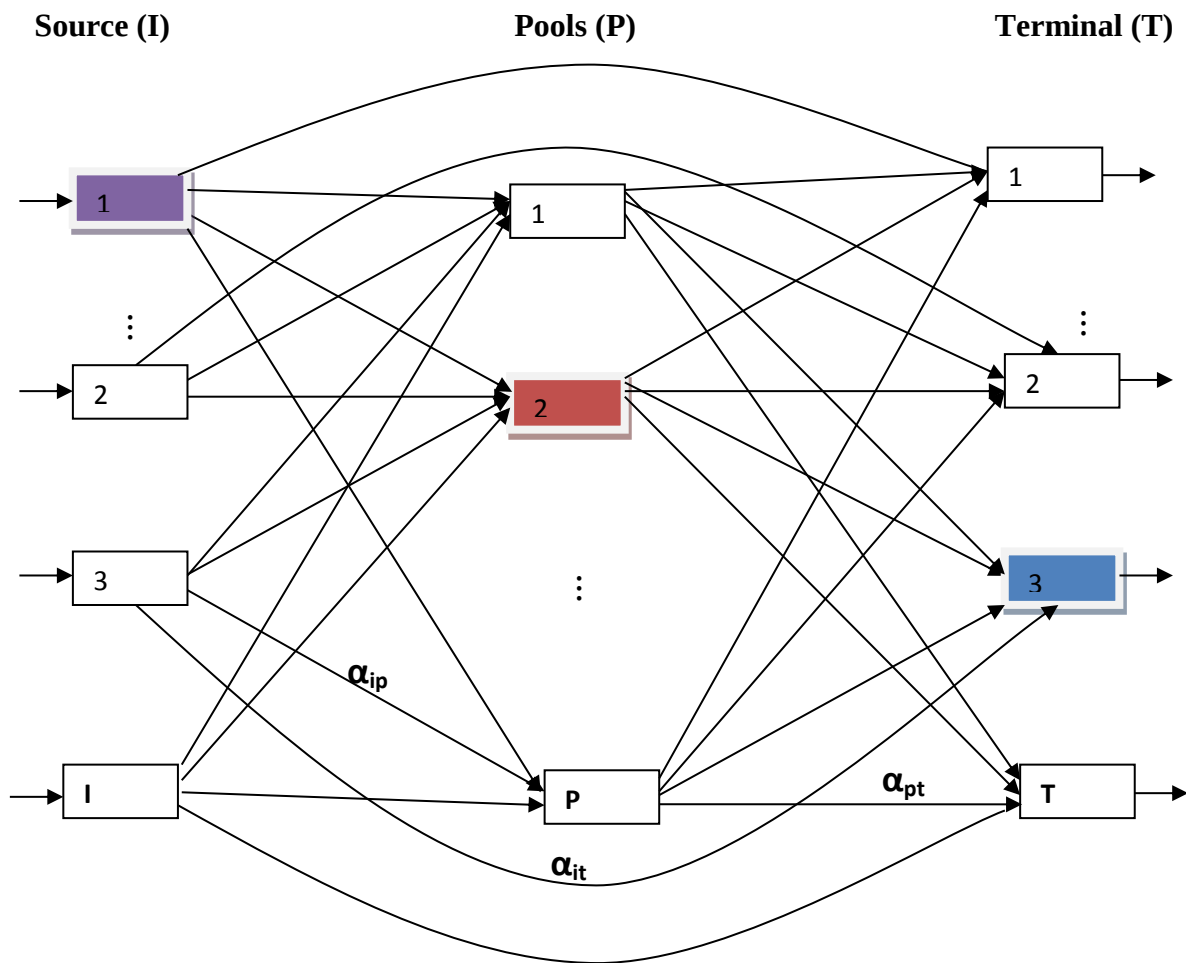


Fig 1: Standard Pooling Problem

Indicatively from the figure above, the figure is a network structure that illustrates the flow interaction of products from the source or input (I) nodes to the pool tanks (P) where mixing is done before it's further transported to the terminal or output (T). Something is interesting about the whole network and that is the cyclic or acyclic nature of the network. Cyclic because it can take the bypass trend, that is movement from the input nodes to the terminals and acyclic because it follows the path, $i \rightarrow p \rightarrow t$, that is from source to pool and down to terminal.

Notations Used in this study

$i \in I$ raw materials or feeds or inputs, $I = \{1, \dots, |I|\}$

$p \in P$ pool, $P = \{1, \dots, |P|\}$

$t \in T$ terminal or outputs or blend, $T = \{1, \dots, |T|\}$

$k \in K$ quality attribute, $K = \{1, \dots, |K|\}$

α_{ip} total flow of material from i^{th} input p^{th} pool

α_{pt} total flow of material from p^{th} pool to t^{th} terminal

α_{it} total flow of material from i^{th} input to t^{th} terminal

f_{ip} proportion of flow from input i to pool p

f_{pt} proportion of flow from pool to terminal

C_i unit cost of raw material or input i

d_t price of final blend or output t

C_{ip} unit cost of transporting feed i to pool p

C_{pt} unit cost of transporting material from pool p to terminal t

C_{it} unit cost of bypass transportation of material from input i to the terminal t

b_i feed or input capacity

b_p pool capacity

b_t product or output capacity

q_i^k k quality bound at node i or input i

q_t^k quality bound at terminal t

Suppose there exist a set of three disjoint nodes I, P and T denoted by θ . Also if there exist a set of arc denoted by Φ such that for a directed acyclic graph $\Omega(\theta, \Phi)$ where $\theta = I \cup P \cup T$.

For any node $m \in \theta, n \in \theta$, let $\theta_m^+ = \{(m, n) \in \Phi\}$ and $\theta_n^- = \{(n, m) \in \Phi\}$ be the sets of out and in-neighbours of m and n respectively.

Let I and T be non-empty where I and T are the set of sources or feed and terminals or outputs respectively. Consider the assumption also, that $\Phi(I \times P) \cup (I \times T) \cup (P \times T)$, Indicating θ_i^- is empty $\forall i \in I$ and θ_t^+ is also empty $\forall t \in T$. Let $Q \subseteq I \times P \times T$ represent the set of paths involving dual arcs in Ω . We let all nodes in P as pools or intermediate tanks where mixing takes place. We refer K as a finite set of quality attribute which implies that $K = \{1, 2 \dots k\} \forall k \in K$. Also for every i , we define q_i^k as the parameter depicting the quality attribute k at feed $i \in I$. It is necessary to relate a real constant with every source (I) and each terminal (T) as can be seen in the case of q_i^k and q_t^k where q_t^k is the quality capacity of attribute K at terminal $t \in T$.

If for every $i \in \theta$, we denote b_i as the constant flow capacity and for every arc $(m, n) \in \Omega$, we denote the constant unit cost as C_{mn} .

Given that f_{ip} is the flow proportion via pool $p \in P$ derived from input $i \in I$. If the flow via p is non-zero, then

$$f_{ip} = \frac{\alpha_{ip}}{\sum_p \alpha_{pt}} \quad (1)$$

In same vein, let f_{pt} be the proportion of flow via terminal $t \in T$ derived from pool $p \in P$. If the flow via t is non-zero, then

$$f_{pt} = \frac{\alpha_{pt}}{\sum_i \alpha_{ip}} \quad (2)$$

More so, define y_{ipt} as the flow along the path i, p and $t \quad \forall (i, p, t) \in (I \times P \times T) / P$.

Equations (1) and (2) yield (3) by multiplication

$$\alpha_{ip} \alpha_{pt} = f_{ip} f_{pt} \sum_i \sum_t \alpha_{ip} \alpha_{pt} \quad (3)$$

Hence, the new hybridized model is given thus;

$$NHYB^- = \text{Min}_{f, \alpha, y} \sum_i \sum_t C_{it} \alpha_{it} + \sum_i \sum_p \sum_t C_{ip} y_{ipt} + \sum_i \sum_p \sum_t C_{pt} y_{ipt} \quad (4)$$

$$\text{s.t} \quad \sum_t \left(\alpha_{it} + \sum_p y_{ipt} \right) \leq b_i, \quad i \in I \quad (5)$$

$$\sum_i \sum_t y_{ipt} \leq b_p, \quad p \in P \quad (6)$$

$$\sum_t \left(\alpha_{it} + \sum_p y_{ipt} \right) \leq b_t, \quad t \in T \quad (7)$$

$$q_i^k \sum_i \sum_p y_{ipt} + q_i^k \sum_i \alpha_{it} \leq q_t^k \sum_i \sum_p y_{ipt} + q_t^k \sum_i \alpha_{it}, \quad t \in T, \quad k \in K \quad p \in P \quad (8)$$

$$\alpha_{ip} \alpha_{pt} = f_{ip} f_{pt} \sum_i \sum_t \alpha_{ip} \alpha_{pt} \quad p \in P \quad (9)$$

$$\sum_i q_i^k f_{ip} - \sum_t q_t^k f_{pt} \leq 0 \quad p \in P \quad (10)$$

$$\sum_i q_i^k f_{it} - \sum_t q_t^k f_{pt} \leq 0 \quad p \in P \quad (11)$$

$$\sum_p q_p^k f_{ip} - \sum_t q_t^k f_{pt} \leq 0 \quad i \in I \quad (12)$$

$$\sum_i f_{ip} = 1 \quad p \in P \quad (13)$$

$$\sum_i y_{ipt} = \alpha_{pt} \quad p \in P, t \in T \quad (14)$$

$$y_{ipt} = \alpha_{pt} f_{ip} \quad (i, p, t) \in P \quad (15)$$

$$\sum_t f_{pt} = 1 \quad p \in P \quad (16)$$

$$\sum_t f_{pt} = 1 \quad p \in P \quad (17)$$

$$\sum_i f_{ip} - \sum_t f_{pt} = 0 \quad p \in P$$

$$y_{ipt} = \alpha_{pt} \sum_i f_{ip} \quad (i, p, t) \in P \text{ Since } \sum_i f_{ip} = 1 \quad p \in P \quad (18)$$

$$f_{pt} \geq 0, \quad p \in I \cup P, \quad t \in T \quad (19)$$

$$b_t \leq b_p \leq b_i \quad (20)$$

$$0 \leq f_{ip}, f_{pt} \leq 1 \quad p \in P, i \in I, t \in T \quad (21)$$

2.1 IMPLICATIONS/INTERPRETATIONS OF THE NHYB- MODEL

The objective function (4) shows the optimization of the total cost of transporting the product to the destination. Constraints (5) to (7) show the input, pool and terminal capacities respectively. Constraint (8) shows the quality at the input with that at the terminal. Equation (9) shows the strength of the two proportions (input and terminal) which allows for larger instances. Constraints (10) to (12) show the quality bounds at the inputs, pools and terminals respectively. Constraints (13) and (16) clearly show that the sum of the flow proportion from source to pool and from pool to terminal, both give an assumed value of 1. Constraint (17) shows the equality in the proportions which must be satisfied. The interaction of constraints (13) and (16) gives (17). Constraint (13) explains that the flow along the route (i, p, t) equals the flow along arc (p, t) . Constraint (17) explains the flow along the route (i, p, t) equals the product of the flow along arc (p, t) and the flow from pool p arising from input i . Constraint (20) explains the boundary condition of the proportions. Every other constraint is a necessary redundant cut that will add to the performance of the model.

Overall, our model seems to be a challenging optimization issue requiring a lot of computational resources and knowledge to solve. The relaxation concept could be applied to reduce expenses while optimizing the transportation of a product from an input to a terminal destination.

An optimization model that can be used to improve the material flow in a production process with many inputs and outputs is the NHYB-model, which was previously presented. While taking into account quality restrictions and capacity restrictions of the inputs, pools, and terminals, the model aims to reduce the overall cost of transferring materials from the inputs to the outputs.

The model can offer an ideal solution that saves transportation costs while maintaining product quality and conforming to capacity limits by taking into account the movements of materials between inputs, pools, and terminals.

The NHYB-model is adaptable and may be tailored to different production methods and business sectors. The model's parameters and constraints can be changed to accommodate various circumstances and constraints, making it a flexible tool for optimization.

2.2 LINEAR RELAXATION OF THE NHYB- FORMULATION

Subsequent upon the NHYB formulation in this work, we go in search of an expedient solution technique that can help to reduce the complexity of the problems such that they can be handled flexibly for optimal solution and also obtain very tight lower bounds through the relaxation and new branching technique by Alfaki (2013). This is so necessary because of the NP –hardness of the problem and large instances being studied. Hence, reducing the problem to polynomial time $O(n^k)$, where n is the input size, $k = \text{constant}$, within the shortest possible time frame.

Possible Cases Arising From the Boundary Conditions of the Linear Relaxation of NHYB-

$$1. \quad (f - f_-)(\alpha - \alpha_-) = 0 \quad (22)$$

$$2. \quad (f - f^-)(\alpha - \alpha^-) = 0 \quad (23)$$

$$3. \quad (f - f_-)(\alpha - \alpha^-) = 0 \quad (24)$$

$$4. \quad (f - f^-)(\alpha - \alpha_-) = 0 \quad (25)$$

Where α is the total flow of product, f is the Proportion of flow, f_- is the lower bound of the proportion variable, f^- is the upper bound of the proportion variable, α_- is the lower bound of the total flow of product, α^- is the upper bound of the total flow of product

The expansion of the above equations (22) to (25) results to the following.

$$f\alpha = f\alpha_- + f_- \alpha - f_- \alpha_- \quad (26)$$

$$f\alpha = f\alpha^- + f^- \alpha - f^- \alpha^- \quad (27)$$

$$f\alpha = f\alpha^- + f_- \alpha - f_- \alpha^- \quad (28)$$

$$f\alpha = f\alpha_- + f^- \alpha - f^- \alpha_- \quad (29)$$

Thus a linear relaxation of (4) to (21) is obtained by bounding Y_{ipt} in between the concave and convex envelopes of $\alpha_{ip}f_{pt}$ for all $i, p, t \in P$. Again, for all $p, t \in PXT$, let f_{-pt} and f_{+pt} be the lower and upper bound of f_{pt} respectively. Similarly, let α_{-ip} and α_{+ip} be the lower and upper bounds of the variable α_{ip} respectively. We can find the finite bounds wherein any optimal value of α_{ip} lies. For instance, it is constructive to put $\alpha_{-ip} = 0$ and $\alpha_{+ip} = \min\{b_i, b_p\}$ or $f_{-pt} = 0$ and $f_{+pt} = \min\{b_p, b_t\}$.

In line with the logic of *convex and concave envelopes* (Al-khayyal and Falk 1983; McCormick 1976), it can be shown from eqns (22) – (29) that the *concave envelopes* of $f\alpha$ on the rectangle $\tilde{D} = [f_{-}, f_{+}] \times [\alpha_{-}, \alpha_{+}]$ are

$$Vex_c(f\alpha) = \max\{f\alpha_{+} + f_{-}\alpha - f_{-}\alpha_{-}, f\alpha_{-} + f_{+}\alpha - f_{+}\alpha_{-}\} \quad (30)$$

and

$$Cav_c(f\alpha) = \min\{f\alpha_{+} + f_{-}\alpha - f_{-}\alpha_{-}, f\alpha_{-} + f_{+}\alpha - f_{+}\alpha_{-}\} \quad (31) \text{ respectively.}$$

2.3 BRANCHING APPROACHES

Let (f, α, y) denote the solution to the LP relaxation of the proposed NHYB- formulation of the pooling problems. As earlier defined, let $\tilde{D}_{ipt}$ be the rectangle enclosing the all feasible α_{ip}, f_{pt} . Let $\tilde{D}_{tip}$ represent the axis parallel to the rectangle enclosing a subset of feasible values of (f_{ip}, α_{pt}) .

Giving consideration to a node in the search tree where the LP relaxation given by the set of rectangles $\{\tilde{D}_{ipt} : (i, p, t) \in p\}$ and or $\{\tilde{D}_{tip} : (i, p, t) \in p\}$ apply and has the optimal solution (f, α, y) . There exist also infeasible measures defined for all $(i, p, t) \in p$ as

$$\tilde{h}_{pt}^i = \max\{f_{pt}, \alpha_{ip} - Vex_{\tilde{D}_{tip}}(f_{pt}, \alpha_{ip}), Cav_{\tilde{D}_{tip}}(f_{pt}, \alpha_{ip}) - f_{pt}, \alpha_{ip}\} \quad (i, p, t) \in p \quad (32)$$

$$\tilde{h}_{ip}^t = \max\{f_{ip}, \alpha_{pt} - Vex_{\tilde{D}_{tip}}(f_{ip}, \alpha_{pt}), Cav_{\tilde{D}_{tip}}(f_{ip}, \alpha_{pt}) - f_{ip}, \alpha_{pt}\} \quad (i, p, t) \in p \quad (33)$$

We give consideration to two approaches for selecting the variables to branch on the max –infeas approach which is applicable to our NHYB- formulation. We choose some $(i, p, t) \in p$ Maximizing $\tilde{h}_{pt}^i$ and branch on the corresponding variable pair in the NHYB- model or formulation, both $\tilde{h}_{pt}^i$ and $\tilde{h}_{ip}^t$ are well defined and the condition for picking the max-infeas approach is to $\max(\tilde{h}_{pt}^i, \tilde{h}_{ip}^t)$. We then branch on either (f_{pt}, α_{ip}) or (f_{ip}, α_{pt}) depending on whether $\tilde{h}_{pt}^i$ is greater than $\tilde{h}_{ip}^t$ or vice versa.

2.4 MATHEMATICAL PROPERTIES OF THE PROPOSED MODEL

Proposition 1: For any tripartite network of pooling problem, $k \leq q$, where k is the quality of pool l and q is the quality of products $(k \in K, q \in Q)$.

Proof

Assume inputs nodes to be $I_1, I_2, I_3, \dots, I_\alpha$, pools or intermediate tanks to be $P_1, P_2, P_3, \dots, P_\beta$. Similarly, let the products or terminals be denoted by $T_1, T_2, T_3, \dots, T_\gamma$. It is trivial to notice that the contaminants in the inputs/nodes make the quality of the pools greater than the source qualities except for cyclic network, where it becomes a blending problem (from source to the terminal).

$\Rightarrow I_{1g}, I_{2g}, I_{3g}, \dots, I_{\alpha g}, \leq P_{1k}, P_{2k}, P_{3k}, \dots, P_{\beta k}$, where g is the quality (contaminants) of the source nodes and $g \in G$. In order to meet the customers' demand while obeying the upper limit A_α - for component availabilities, D_β -upper limits for product demands, P_β^k -upper limits for pool sizes and T_γ^q for quality of the products, $k \in K, q \in Q$, also assume k as the quality of pool l and q is the quality of products or terminals for $k \in K, q \in Q$

$$\begin{aligned} \sum_k P_{\beta-}^k - \sum_q T_\gamma^q &\leq 0 \\ \Rightarrow \sum_k P_\beta^k &\leq \sum_q T_\gamma^q \end{aligned}$$

Clearly, $k \leq q$ □

Proposition 2: In the realm of the resulting relaxations, the infimum of the new hybridized formulation (infNHYB) is tighter than the infimum of the lower bounds of the TP and the STP counterparts.

Proof

If there exist a set of lower bounds R such that $M \in R$ where R is the set containing M .

Let $M_1, M_2, M_3 \dots$ be the elements of set M , notice that $M_1 < M_2 < M_3 \dots$. Again, there exist linear constraints of the relaxations of formulations TP-, STP- (see Alfaki 2013) and NHYB⁻, however if we denote the linear relaxation constraints of the STP⁻ by $M_{1\zeta}, M_{2\zeta}, M_{3\zeta}, M_{4\zeta} \dots M_{r\zeta}$, and that of NHYB⁻ as $N_{1\zeta}, N_{2\zeta}, N_{3\zeta}, N_{4\zeta} \dots N_{q\zeta}$. It's indicative that all linear constraints of STP are derived by NHYB⁻ from equation (14).

Hence, $M_{1\zeta}, M_{2\zeta}, M_{3\zeta}, M_{4\zeta} \dots M_{r\zeta} < N_{1\zeta}, N_{2\zeta}, N_{3\zeta}, N_{4\zeta} \dots N_{q\zeta}$, where $M, N \in R$.

Moreover, the bilinear constraints:

$$\sum_i q_i^k f_{ip} - \sum_t q_t^k f_{pt} \leq 0 \quad p \in P$$

$$\sum_i q_i^k f_{it} - \sum_t q_t^k f_{pt} \leq 0 \quad p \in P$$

$$\sum_p q_p^k f_{ip} - \sum_t q_t^k f_{pt} \leq 0 \quad i \in I$$

are merits of the NHYB⁻ formulation over STP- formulation which are linearised in the refining process of relaxation. The $\inf(N)$ must be greater than $\inf(M)$ which indicates the tightness of the lower bounds of set M . If we denote $\inf(N)$ by F_N , then $\inf(M) = F_N + \varepsilon$ showing the superiority of NHYB⁻ formulation. Since $F_N \neq F_N + \varepsilon$, therefore, the infimum of NHYB⁻ is greater and by implication, the lower bounds are tighter than its competitor. The proof of the contrast between NHYB⁻ and TP are analogously followed.

Proposition 3: The lower limit of the value of the optimal objective function as obtained from the NHYB⁻ relaxation is strongly tighter than those obtained from its other competitors (TP and STP formulation). $\square$

Proof

It is trivial to observe that the NHYB⁻ bound is strongly tighter than that of the PQ⁻ bound since the NHYB⁻ formulation births every linear constraints of the PQ⁻ formulation and that the relaxation of NHYB⁻ inherits eqn (9). The NHYB⁻ therefore, superimposes its competitors (TP and STP) is analogously proven $\square$.

3. COMPUTATIONAL ANALYSIS AND DISCUSSION

3.1 TEST OF THE FOURTEEN (14) SAMPLES

To illustrate the flexibility, robustness and smart nature of the proposed new hybridised model (NHYB-), we subject some dataset to our models using software called General Algebraic Modelling System (GAMS). All analyses given in this work were performed on a well advanced computer. Again, the link to the dataset used in this work can be provided upon request. For clarity, fourteen instances from the literature were used and fifteen grouped larger instances were simulated to test our model.

We refer to the 14 dataset from the literature by using the notations Adhya1-4 for samples 1-4 in (Adhya et al., 1999), Bental4-5 for samples 4-5 in (Ben-Tal et al., 1994), Foulds2-5 for samples 2- 5, Haverly1-3 for the sample in (Haverly, 1978), and RT2 for the samples with the same indications in (Audet et al., 2000). We shall compare our results with the results of PQ and STP as given in Alfaki and Haugland (2013).

Table 3.1: Size of the fourteen standard test samples.

Samples	I	P	T	k	PQ			STP/NHYB		
					Vars	nlts	lcs	Vars	nlts	lcs
Adhya 1	5	2	4	4	33	20	42	46	40	57
Adhya 2	5	2	4	6	33	20	50	46	40	65
Adhya 3	8	3	4	6	52	32	62	72	64	85
Adhya 4	8	2	5	4	58	40	55	76	80	75
Bental 4	4	1	2	1	13	6	15	18	12	21
Bental 5	5	3	5	2	92	60	53	119	120	83
Foulds 2	6	2	4	1	36	16	30	48	32	44
Foulds 3	11	8	16	1	672	512	219	832	1024	387
Foulds 4	11	8	16	1	672	512	219	832	1024	387
Foulds 5	11	4	16	1	608	512	147	704	1024	247
Haverly 1	3	1	2	1	10	4	13	14	8	18

Haverly 2	3	1	2	1	10	4	13	14	8	18
Haverly 3	3	1	2	1	10	4	13	14	8	18
RT 2	3	2	3	8	34	18	46	46	36	60

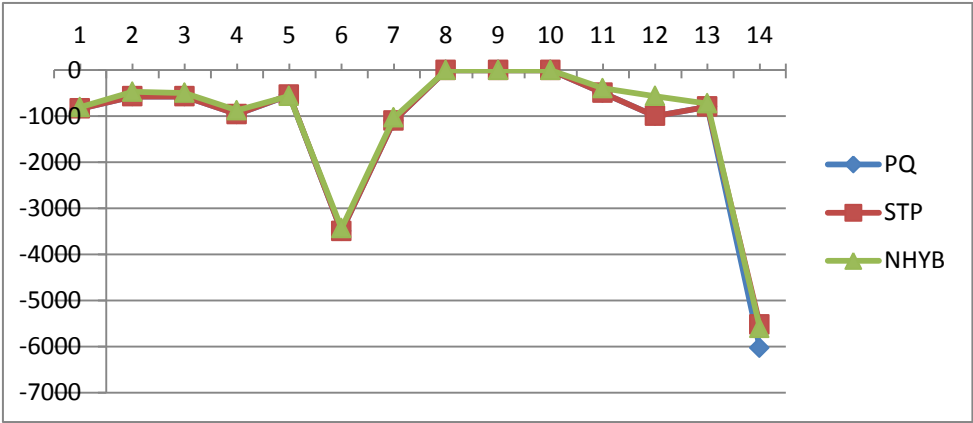
The sample identifier and size with respect to the $|I|$, $|P|$, $|T|$, and $|K|$ are provided in the first five columns of Table 3.1. Following columns provide the size of variables, different bilinear terms, and linear constraints for each formulation applied to each problem sample.

Table 3.2: Presentation of the Relaxation Values.

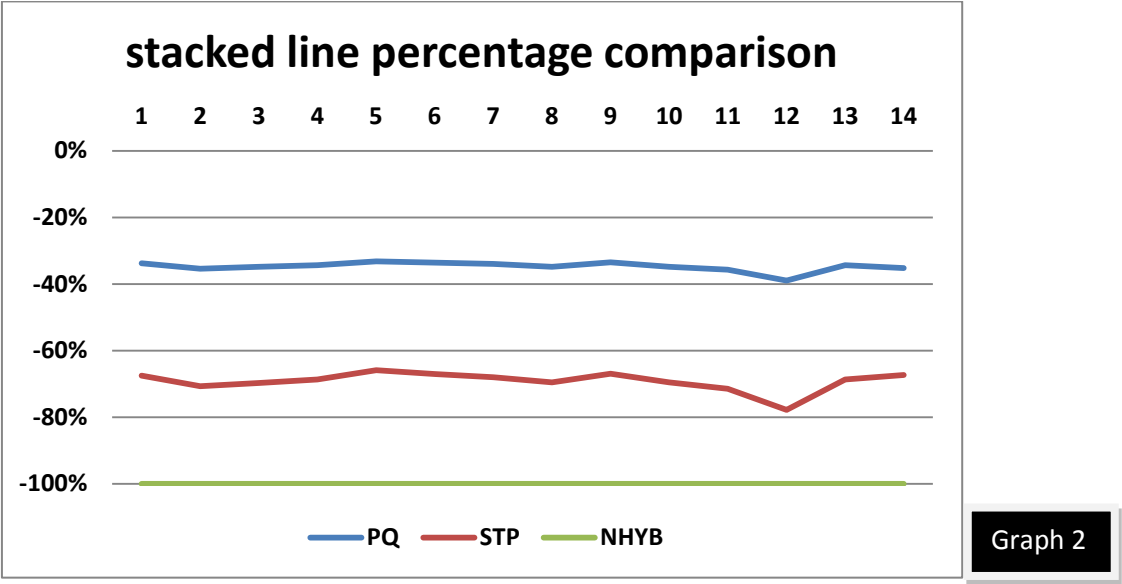
Samples	Values of the Objective function of the relaxations			
	PQ-relax	STP-relax	NHYB-relax	Global optimality
Adhya 1	-840.27	-840.27	-810.17	-549.80
Adhya 2	-574.78	-574.78	-476.19	-549.80
Adhya 3	-574.78	-574.78	-561.05	-561.05
Adhya 4	-961.93	-961.93	-678.64	-450.00
Bental 4	-550.00	-541.67	-565.87	-450.00
Bental 5	-3500.27	-3500.27	-3500.27	-3500.27
Foulds 2	-1100.00	-1100.00	-1100.00	-1100.27
Foulds 3	-8.00	-8.00	-8.00	-8.00
Foulds 4	-8.00	-8.00	-8.00	-8.00
Foulds 5	-8.00	-8.00	-8.00	-8.00
Haverly 1	-500.00	-500.00	-400.00	-400.00
Haverly 2	-1000.00	-1000.00	-568.97	-600.00
Haverly 3	-800.00	-800.00	-730.77	-750.00
RT 2	-6034.87	-5528.25	-5600.65	-4391.83

Discussion 1

The chart demonstrates that in ten situations, Adhya1, Adhya3, Adhya4, Bental5, Foulds2, Foulds3, Foulds4, Haverly1, Haverly2 and Haverly3and RT2, although, for Bental5, Foulds2, Foulds3, Foulds4, the performance was competitive (equal performance), observe that there is an enhancement of the lower bound of the NHYB-relaxation. The closer the relaxation values to the global optimal results, the better the solution. Additionally, the NHYB-relaxation outperforms the, PQ and the STP-relaxations in almost all cases. Although, the STP-relaxation performed better in the instances of Bental4 and RT2, however, the best bound obtained by either the PQ- or STP-relaxation is, however, level with the NHYB-bound in all the cases given in Table 3.2. see the graphical depictions of the results in table 3.2 above. Graph 1 shows the scattered plot of the results and graph 2 shows the percentage of the stacked line of the obtained results.



Graph 1



3.2 TEST OF THE SIMULATED LARGER FIFTEEN (15) SAMPLES

There is evidence from experiments that conventional pooling issue examples from the literature may be resolved quickly. Experiments with bigger samples are required to produce more interesting assessments of the formulation proposed in this study. We created a set of 15 instances at random and separated them into three equal groups (P, Q, and R), with each group's examples having an equal number of sources, pools, terminals, and quality characteristics. Within a group, there are variations in the number of arcs. Considering a probability, referred to as the expected or predicted network density, also defined as input to the sample creation mechanism, we denote an arc (i,j), where $(i,j) \in (I \times P) \cup (I \times T) \cup (P \times T)$. Table 4.5 provides information on network sizes and the range of anticipated densities for each category.

All samples are created with each attribute at the terminals having both an upper and a lower quality constraint. We provide the number of variables (vars), bilinear terms (nlts), and linear constraints (lcs) for each formulation for each sample in the larger-scale collection in Table 4.6. The group to which the instance belongs is indicated by the first character in the instance identifier shown in this table.

Table 3.3: Description of larger scale samples

Group	# samples	Size if instances in group				Boundary of anticipated density
		I	P	T	K	
P	5	60	30	45	42	0.30 - 0.80
Q	5	105	51	63	34	0.30 - 0.80
R	5	180	45	150	40	0.30 - 0.80

Table 3.4: The size of the larger-scale samples.

Samples	STP/NHYB		
	Vars	nlts	lcs
P1	767	678	768
P2	987	698	987
P3	1167	1685	1012
P4	1543	1786	1321
P5	1855	1987	1434
Q1	3424	2314	1546
Q2	4420	3562	1662
Q3	4879	7689	2321

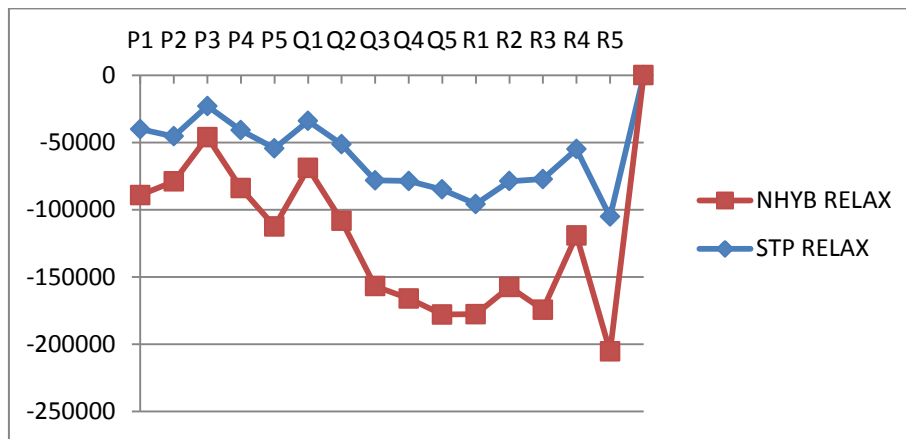
Q4	8900	16986	4326
Q5	12976	19875	7433
R1	14765	20970	8902
R2	18342	29874	9675
R3	20441	36589	9873
R4	21876	41765	10897
R5	23879	53566	11313

Table 3.4 presents the findings of the computational trials of table 3.3. The instance identifier is given in the first column; the optimal objective function value across three relaxations is given in columns 2-4

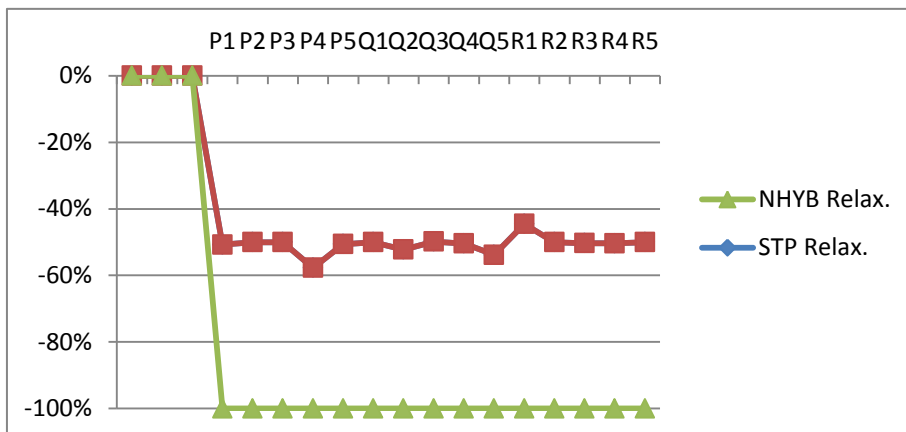
Table 3.5: Presentation of the Relaxation Values for the Simulated Samples

Samples	Objective function values of the relaxations	
	STP-relax	NHYB-relax
P1	-40165.56	-49023.16
P2	-45467.83	-33545.76
P3	-23023.17	-23023.17
P4	-41001.65	-43034.54
P5	-54350.05	-58109.77
Q1	-34018.54	-35008.61
Q2	-51345.11	-56920.13
Q3	-78165.34	-78761.99
Q4	-78631.33	-87432.86
Q5	-85026.59	-92993.18
R1	-95786.88	-81949.64
R2	-78761.99	-78761.99
R3	-77218.03	-97227.21
R4	-54982.46	-64170.65
R5	-105230.10	-100230.05

From the above table 3.5, it is clear that the NHYB-formulation performed better with the resultant greedy lower bound values for at least eight (8) larger instances (P1, P4, P5, Q1, Q2, Q5, R3, R4) and equalled with the STP- formulation in about two (2) larger instances (R2 and P3). In nutshell, NHYB model overrides the STP model with the eight best performances as against the three best performances by the STP (R5, P2 and R1).



Graph 3



Graph 4

Clearly, the NHYB formulation efficiently outperformed the other formulations; also showing its superiority over others. Notice that in graph 4, the NHYB model lies on the 100% showing its relative superior performance. In same vein, graph 3 shows the scattered plot which demonstrates the strength of the lower bound of the models.

4. CONCLUSION

We developed a model (NHYB) that has the tenacity to accommodate large instances of pools and robustly improve the quality of the products while aiming at optimising the cost of transportation. The developed models were largely tested with simulated and literature data.

This study is significant because it will help to address the identified problems in our petroleum industries which by implication will guide our industries on better production and maximum profit. Again, from the results of this study, it is necessary to advise the petroleum industries to resort to pooling or blending since it is cheaper and would save cost of production. The pooling here simply means physical mixture of different crude or gas with varying quality attributes which after undergoing correct mixture based on the model will yield the expected better result. In the pooling process, the attention is to reduce the level of contaminations or sulphur content of the products under mixing experiment. Importantly, the model also helps to build an optimal network of transportation of the products such that the cost of transportation is minimized, thereby saving more income.

References

- Adhya, N., Tawarmalani, M. and Sahinidis, N.V.,(1999) A Lagrangian approach to the pooling problem. *Ind. Eng. Chem. Res.*, 38(5):1965-1972.
- Alfaki, M., and Haugland, D. (2013). Strong formulations for the pooling problem. *Journal of Global Optimization*. doi:10.1007/s10898-012-9875-6
- Al-Khayyal, F.A. and Falk, J.E (1983). Jointly constrained biconvex programming. *Math. of Oper. Res.*, 8(2): 273-286.
- Armagan, (2009) A new Lagrangean approach to the pooling problem. *J. of Glob. Optim.*, 9(2) : 56 -69
- Audet, C., Brimberg, J., Hansen, P., Le Digabel, S. and Mladenovic, N.(2004), Pooling problem: Alternate formulations and solution methods. *Manag. Sci.*, 50(6):761-776.
- Audet, C., Hansen, P., Jaumard, B. and Savard, G. (2000), A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. *Math. Program.*, 87(1):131-152.

7. Ben-Tal, A., Eiger, G. and Gershovitz, V (1994), Global minimization by reducing the duality gap. *Math. Program.*, 63:193–212.
8. Chen Y., Christos T.M and Xiaomin Z. (2022), Tightening discretization-based MILP models for the pooling problem using upper bounds on bilinear terms, *J. of Glob. Optim.*, 15(2): 432-443
9. D'Ambrosio M.A., Janak, S.L. and Floudas, C.A.(2011), Continuous-time models for short-term scheduling of multipurpose batch plants: A comparative study. *Ind. Eng. Chem. Res.*, 45:6190-6209.
10. Haverly, C.A.(1978), Studies of the behavior of recursion for the pooling problem. *ACM SIGMAP Bulletin*, 25:19 – 28.
11. Lennart, C., Lübbecke, M., Möhring, R., & Stiller, S. (2012). The concept of recoverable robustness, linear programming recovery.
12. McCormick, G.P.(1976), Computability of global solutions to factorable nonconvex programs: Part 1-convex underestimating problems. *Math. Program.*, 10(1):147-175.
13. Meyer, C.A. and Floudas, C.A.(2006), Global optimization of a combinatorially complex generalized pooling problem. *AIChE J.*, 52(3):1027 -1037.
14. Michal and Rafal (2023). Social network analysis and mining. *Springer link*, 17(1), 77-95
15. Misener, R, and Tawarmalani, M. (2010). Accelerating branch-and-bound through a modeling language construct for relaxation-specific constraints. *Journal of Global Optimization*, 32 (2), 259–280.
16. Misener, R., and Floudas, C.A. (2009). Advances for the pooling problem: Modeling, global optimization, and computational studies. *Applied and Computational Mathematics*, 8 (1), 3–22.
17. Mostafa, E. M. E (2008). First-order penalty methods for computing suboptimal output feedback controllers. *Appl. Comput. Math.*, 7(1):66 - 83.
18. Nishi, T.: A semidefinite programming relaxation approach for the pooling problem. Master's thesis, Department of Applied Mathematics and Physics, Kyoto University (2010). URL <http://www-optima.amp.i.kyoto-u.ac.jp/result/masterdoc/21nishi.pdf>
19. Quesada, I. and Grossmann, I.E.(1995), Global optimization of bilinear process networks with multicomponent flows. *Comput. & Chem. Eng.*, 19:1219-1242.
20. Radu and Misener (2018), piecewise parametric structure in the pooling problem: from sparse strongly-polynomial solutions to NP-hardness. *J. Of Glob Optim.*, 17(4), 577 - 591
21. Ruiz, M.H., Chung, H.S. and Marshall, J.F.(2013), A "transformation" method for calculating the research and motor octane numbers of gasoline blends. *Ind. Eng. Chem. Fundamen.*, 20(3):195 -204.
22. Serali, H.D.and Alameddine, A.(1992), A new reformulation-linearization technique for bilinear programming problems. *J. of Glob. Optim.*, 2:379-410.
23. Tawarmalani, M. and Sahinidis, N.V.(2002), *Convexiflcation and Global Optimization in Continuous and MixedInteger Nonlinear Programming: Theory, Applications, Software, and Applications*. Nonconvex Optimization and Its Applications. Kluwer Academic Publishers, Norwell, MA, USA.
24. Visweswaran,V , Floudas and P. M. Pardalos (2009),, MINLP: Applications in blending and pooling. In C. A. editors, *Encyclopedia of Optimization*, pages 2114-2121. Springer Science, 2nd edition.