



Canonical partition by covering and clopen homeomorphism

Anima Nuda 

Retired, independent scholar.

*Corresponding Author

Anima Nuda

Email:

anima.nuda1944@gmail.com

Article History

Received: 03.06.2026

Accepted: 24.07.2026

Published: 13.08.2026

Abstract: Each covering induces a partition, such that every cover is union of partition classes, and it's the coarsest between all partitions. This theorem is intuitively evident while the demonstration is not immediate.

Moreover any partition is topological base of cl-open topology, that is homeomorphic to some discrete topology.

Keywords: covering, partition, equivalence class, saturation, cl-open topology, a-discrete topology.

2010 AMS subject classification: 03E02, 03E47, 54A05, 54A10.

This work is licensed under a [Creative Commons Attribution-NonCommercial 4.0 International License](https://creativecommons.org/licenses/by-nc/4.0/) (CC BY-NC 4.0)

1. Introduction

In set theory the generic **coverings** mainly intervene in the compactness (topology) study. For the rest their role is marginal, while greater importance belongs to particular coverings called **partitions**. Each *covering* canonically induces a *partition*, while the *vice versa* not hold. The transformation of the coverings into partitions is intuitive and immediate, apparently self-evident: however this is a topic *ignored* in the literature. Here we limit ourselves to illustrating the transformation from covering to partitioning.

In the second part we mention the *partition topologies*, called **clopen** or **quasi-discrete** [1]. An homeomorphism theorem is proposed which also clarifies the *quasi-discrete* appellation.

2. Premise

2.1. Assumptions

Ω	environment set,
$P(\Omega)$	its power set,
$\{x\}_{x \in X} = X \in P(\Omega)$	a set with its elements,

Θ	index set ($\theta \in \Theta$),
$\#A$	cardinality of set A,
$\ni$	such that.

2.2. Preliminaries

2.2.1. Coverings

$\emptyset \notin C \subseteq P(\Omega), \cup C = \Omega :$	
$C = C_\Omega = \{C_\theta\}_{\theta \in \Theta}$	covering of Ω ,
$C_\theta = C \in C$	cover or covering-element .
$C_x = \{C \in C \mid x \in C\}$	coverage (or <i>nhb</i>) of x,
$C_x = \cap C_x$	hull (or <i>minimal nhb</i>) of x,
$C_X = \{C_x\}_{x \in \Omega}$	new narrow covering by C.

2.2.2. Partitions

$\emptyset \notin \Pi = \{P_\theta\}_{\theta \in \Theta} \subseteq P(\Omega), \cup \Pi = \Omega \ \& \ \forall \theta$ $, \eta \in \Theta \exists \theta \neq \eta \Rightarrow P_\theta \cap P_\eta = \emptyset :$	
$C_\Pi = \Pi = \{[x]_{x \in \Omega}\}$	partition-covering
$[x]$	equivalence class of x in Π ,

Cite this article:

Nuda, A., (2026). Canonical partition by covering and clopen homeomorphism. *ISAR Journal of Science and Technology*, 4(8), 1-3.

$$E = [x] \in C_\Pi$$

tile or *partition-element*.

3. Partitions

3.1. Equivalence Relations

A relation R on Ω (with the *diagonal* $\Delta_X = \{(x, x)\}_{x \in \Omega}$), *reflexive* ($\Delta_X \subseteq R$), *symmetric* ($R = R^{-1}$) and *transitive* ($R \circ R \subseteq \Delta_X$), is called **equivalence relation** R on Ω and, $\forall x \in \Omega$, the set $[x] = \{y \in \Omega \mid xRy\}$ is called *equivalence class* of x . The set $\{[x]\}_{x \in \Omega}$ is a *partition* of Ω ; and given a partition $\Pi = \{P_\theta\}_{\theta \in \Theta}$ of Ω , we define the equivalence relation R : xRy iff $(x \in P_\theta, y \in P_\eta \Rightarrow \theta = \eta)$.

3.2. Saturation

Let C be a covering of Ω and Π_Ω a partition of Ω . Whether for any $C \in \mathcal{C}$ there is a family Φ of Π_Ω such that $C = \cup \Phi$, we call: Π_Ω *saturates* C or C *saturated* by Π_Ω or *saturation* of C with Π_Ω .

3.3. Comparisons

Let Π_1 and Π_2 be partitions on Ω . In the struct $[1]$ ($\Pi_\Omega, \preceq_\Pi$) we define a pre-order relation, denoted $\Pi_1 \preceq \Pi_2$, and we say Π_1 is **finer** (or *stronger* or *larger*) than Π_2 or Π_2 is **coarser** (or *weaker* or *smaller*) than Π_1 , when Π_2 is saturated by Π_1 . For any partition Π it is:

$$\Pi_\Omega = \{\{x\}\}_{x \in \Omega} \preceq \Pi \preceq \{\Omega\} = \Pi_0$$

(Π_0 *indiscrete conventional partition* (<only the environment set>), Π_Ω *discrete partition* (<all singletons>)).

4. Partition from covering

Proposition 1. With previous (2.2.1 hull) denotations, each covering C of Ω generates the new **narrow** covering $C_\Omega = \{C_x\}_{x \in \Omega}$, from which descends a new **partition-covering** of Ω which is the **coarsest saturation** Π_C of Ω (that we call *saturation induced by* (or *associated with*) a covering).

Proof. [2] Arbitrarily chosen a point $x \in \Omega$, the *hull* C_x is *unique* and the *narrow covering* is uniquely determined by construction. Then we define:

$$P_x = \{y \in C_x \mid C_y = C_x\}, \quad N_x = \{y \in C_x \mid C_y \neq C_x\}.$$

It results:

$$P_x \cup N_x = C_x, \quad P_x \cap N_x = \emptyset \Rightarrow [x] = P_x \quad (\{[x]\}_{x \in \Omega} = \{P_x\}_{x \in \Omega} = \Pi_C)$$

and

$$\alpha) \cup_{x \in \Omega} P_x \supseteq \Omega; \quad \beta) [x] \neq [y] \Rightarrow [x] \cap [y] = \emptyset.$$

The $[x]$ construction *acquires* each single *hull* C_x in its *entirety* or **otherwise** the construction *partitions* it into its disjoint subsets and this way for each $x \in \Omega$. ■

5. Topology

5.1. Topology from covering

Proposition 2. From each covering C of Ω there is the *coarsest* topology τ such that contains C called *generated* by C .

Proof. [3] (Ch. I-3.5). ■

5.2. Topology from partition

A topology $\tau \subseteq P(\Omega)$ (and $B(\tau)$ its basis) is called **cl-open** (or **quasi-discrete** [1] or *pseudo-discrete* or **α -discrete**) *topology* when the only open sets of τ are closed (and all the closed sets for τ are open).

Each partition $\Pi = B(\tau_\Pi)$ is basis for a *cl-open* topology, denoted τ_Π :

$$A \subseteq P(\Pi), A \subseteq \tau_\Pi, \quad A = \{CA \in P(\Pi) \mid A \in A\} \Rightarrow A \subseteq \tau_\Pi;$$

$$\tau \subseteq \sigma \Rightarrow \sigma \subseteq \tau \quad (H \in \sigma \Rightarrow H \in \tau \quad (\tau \subseteq \sigma) \Rightarrow H \in \sigma \Rightarrow H \in \tau);$$

$$\{\emptyset, \Omega\} \subseteq \tau \cap \sigma = \tau = \sigma.$$

The next proposition characterizes the *quasi-discrete* topologies and justifies their naming.

5.3. α -discrete topology

Proposition 3. Whether, in a topological space (Ω, τ) , it is $\tau = \sigma$ (with σ its closed set family) then (Ω, τ) is in **homeomorphic correspondence** to $(Y, P(Y))$ for some set Y , and τ is called **α -discrete** topology, where it is $\alpha = \#Y$.

Proof. [4] $\forall x \in \Omega$ let $A_x = \{A \in \tau = \sigma \mid x \in A\}$ be and $A_x = \cap A_x \in \tau = \sigma$. It's: $\forall y \in A_x \Rightarrow A_y = A_x$ ($A_y \subseteq A_x; A_x \setminus A_y \in A_x \Rightarrow A_x \subseteq A_y$). Let $x \approx y \Leftrightarrow A_x = A_y$ be. This $\approx$ is an equivalence relation. $\{[x]_\approx\}_{x \in \Omega}$ is a base for τ and a partition of Ω . $Y = \Omega / \approx$ complete the proof. ■

6. Examples

6.1. Denotations

$\mathbb{R}$ real (r) number set, $\mathbb{R}^- =]-\infty, 0[$ (negative r.), $\mathbb{R}^+ =]0, +\infty[$ (positive r.), $\mathbb{R}_0^- =]-\infty, 0]$ (non-positive r.), $\mathbb{R}_0^+ =]0, +\infty[$ (non-negative r.), $[-n, n] = \{x \in \mathbb{R} \mid -n \leq x \leq n\}$ closed real interval, $\mathbb{R}^{-1/1} =]-1, +1[$ (open relative unitary interval).

6.2. Coverings

$$\alpha) C_\alpha = \{\mathbb{R}^-, \mathbb{R}^+\}$$

$$\beta) C_\beta = \{\mathbb{R}^-, \mathbb{R}^{-1/1}, \mathbb{R}^+\}$$

$$\gamma) C_\gamma = \{\mathbb{R}^-, \mathbb{R}^{-1/1}, \mathbb{R}^+\}$$

$$\delta) C_\delta = \{\mathbb{R}^-, \{0\}, \mathbb{R}^+\}$$

6.3. Partitions from coverings

$$\alpha) \Pi_\gamma = C_{\zeta_a},$$

$$\beta-\gamma) \Pi_\beta = \Pi_\delta = \{]-\infty, -1],]-1, 0[, \{0\},]0, 1[, [1, \infty[\},$$

$$\delta) \Pi_{\zeta_d} = C_\zeta = \{\mathbb{R}^-, \{0\}, \mathbb{R}^+\}.$$

6.4. Topologies from coverings

$$\alpha) \tau_a = \{\emptyset, \mathbb{R}, \mathbb{R}^-, \mathbb{R}^+, \{0\}\}, \sigma_g = \{\emptyset, \mathbb{R}, \mathbb{R}^-, \mathbb{R}^+, \mathbb{R}^- \cup \mathbb{R}^+\};$$

$$\beta) \tau_\beta = \{\emptyset, \mathbb{R}, \mathbb{R}^-, \mathbb{R}^{-1/1}, \mathbb{R}^- \cup \mathbb{R}^+,]1, 0[,]0, 1[,]-1, 0[\cup]0, 1[,]-\infty, 1[,]-1, +\infty[\} (0 \notin \tau_\beta);$$

$$\gamma) \tau_\gamma = \{\emptyset, \mathbb{R}, \mathbb{R}^-, \{0\}, \mathbb{R}^+, \mathbb{R}^{-1/1},]-1, 0[, [0, 1[,]-\infty, 1[,]-1, +\infty[\} (]-1, 0[, [0, 1[\notin \tau_\gamma);$$

$$\delta) \tau_\delta = \{\emptyset, \mathbb{R}, \mathbb{R}^-, \{0\}, \mathbb{R}^+, \mathbb{R}^-, \mathbb{R}^+, \mathbb{R}^- \cup \mathbb{R}^+\}.$$

6.5 Topologies from partitions

In δ we have 3-discrete topology, while in partitions generated by β and γ there is 5-discrete topology. The coverings C_β and C_γ generate different incomparable

topologies τ_β and τ_γ , while they generate the same partition $\Pi_\beta = \Pi_\gamma$, that, in turn, it generates the 5-discrete topology τ_{Π_β} or τ_{Π_γ} strictly finer than τ_β and τ_γ .

7. Conclusions

Here is presented the transformation of an arbitrary covering in partition. The set theory literature, aside from a partial and incomplete proof [5], does not address this topic because, while the statement is self-evident, a proof is not immediately obvious. Furthermore a *homeomorphism theorem* between clopen spaces is presented.

Spontaneity and acknowledgements

The author declares no conflict of interest, no grant received, no data used and he is grateful to the editors for the acceptance.

References

1. Cech, E. (1966), *Topological Spaces* (revised edition by Z.Frolík & M.Katětov), J.Wiley & Sons, N.Y. .
2. Animobono, N.U., Partizione canonica indotta da ricoprimento, *Periodico di Matematica* (submitted).
3. Checcucci, V., Tognoli, A., Vesentini, E. (1969), *Lezioni di topologia generale*, 2^a ed., Feltrinelli Milano,
4. Animobono, N.U., (1993), Topological approach to rough sets, *BUSEFAL* 54: 84-87.
5. Ludewig, M., Partition induced by a cover, Mathoverflow (nov 07, 2024), on line: <https://mathoverflow.net/questions/481954/partition-induced-by-a-cover>